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Direct extraction of one-loop integral coefficients *

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Abstract

We present a general procedure for obtaining the coefficients of the scalar bubble and triangle integral functions of one-loop amplitudes. Coefficients are extracted by considering two-particle and triple unitarity cuts of the corresponding bubble and triangle integral functions. After choosing a specific parameterisation of the cut loop momentum we can uniquely identify the coefficients of the desired integral functions simply by examining the behaviour of the cut integrand as the unconstrained parameters of the cut loop momentum approach infinity. In this way we can produce compact forms for scalar integral coefficients. Applications of this method are presented for both QCD and electroweak processes, including an alternative form for the recently computed three-mass triangle coefficient in the six-photon amplitude $A_6(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$. The direct nature of this extraction procedure allows for a very straightforward automation of the procedure.

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I. INTRODUCTION

Maximising the discovery potential of future colliders such as CERN's Large Hadron Collider (LHC) will rely upon a detailed understanding of Standard Model processes. Distinguishing signals of new physics from background processes requires precise theoretical calculations. These background processes need to be known to at least a next-to-leading order (NLO) level. This in turn entails the need for computation of one-loop amplitudes. Whilst much progress has been made in calculating such processes, the feasibility of producing these needed higher multiplicity amplitudes, such as one-loop processes with one or more vector bosons (W's, Z's and photons) along with multiple jets, strains standard Feynman diagram techniques.

Direct calculations using Feynman diagrams are generally inefficient; the large number of terms and diagrams involved has by necessity demanded (semi)numerical approaches be taken when dealing with higher multiplicity amplitudes. Much progress has been made in this way, numerical evaluations of processes with up to six partons have been performed [1, 2, 3, 4, 5]. On assembling complete amplitudes from Feynman diagrams it is commonly found that large cancellations take place between the various terms. The remaining result is then far more compact than would naively be expected from the complexity of the original Feynman diagrams. The greater simplicity of these final forms has spurred the development of alternative more direct and efficient techniques for calculating these processes.

The elegant and efficient approach of recursion relations has long been a staple part of the tree level calculational approach [6, 7]. Recent progress, inspired by developments in twistor string theory [8, 9], builds upon the idea of recursion relations, but centred around the use of gauge-independent or on-shell intermediate quantities and hence negating a potential source of large cancellations between terms. Britto, Cachazo and Feng [10] initially wrote down a set of tree level recursion relations utilising *on-shell* amplitudes with *complex* values of external momenta. Then, along with Witten [11], they proved these on-shell recursion relations using just a knowledge of the factorisation properties of the amplitudes and Cauchy's theorem. The generality of the proof has led to their application in many diverse areas beyond that of massless gluons and fermions in gauge theory [10, 13]. There have been extensions to theories with massive scalars and fermions [14, 15, 16] as well as amplitudes in gravity [12].

Similarly "on-shell" approaches can also be constructed at loop level. The unitarity

of the perturbative S-matrix can be used to produce compact analytical results by "gluing" together on-shell tree amplitudes to form the desired loop amplitude. This unitarity approach has been developed into a practical technique for the construction of loop amplitudes [17, 18, 19], initially, for computational reasons, for the construction of amplitudes where the loop momentum was kept in $D = 4$ dimensions. This limited its applicability to computations of the "cut-constructible" parts of an amplitude only, i.e. (poly)logarithmic containing terms and any associated π^2 constants. Amplitudes consisting of only such terms, such as supersymmetric amplitudes, can therefore be completely constructed in this way. QCD amplitudes contain in addition rational pieces which cannot be derived using such cuts. The "missing" rational parts are constructible directly from the unitarity approach only by taking the cut loop momentum to be in $D = 4 - 2\epsilon$ dimensions [20]. The greater difficulty of such calculations has, with only a few exceptions [21, 22], restricted the application of this approach, although recent developments [23, 24, 25] have provided new promise for this direction.

The generality of the foundation of on-shell recursion relation techniques does not limit their applicability to tree level processes only. The "missing" rational pieces at one-loop, in QCD and other similar theories, can be constructed in an analogous way to (rational) tree level amplitudes [26, 27]. The "unitarity on-shell bootstrap" technique combines unitarity with on-shell recursion, and provides, in an efficient manner, the complete one-loop amplitude. This approach has been used to produce various new analytic results for amplitudes containing both fixed numbers as well as arbitrary numbers of external legs [28, 29, 30]. Other newly developed alternative methods have also proved fruitful for calculating rational terms [31, 32, 33, 34]. In combination with the required cut-containing terms [35, 36, 37] these new results for the rational loop contributions combine to give the complete analytic form for the one-loop QCD six-gluon amplitude.

The development of efficient techniques for calculating, what were previously difficult to derive rational terms, has emphasised the need to optimise the derivation of the cut-constructible pieces of the amplitude. One-loop amplitudes can be decomposed entirely in terms of a basis of scalar bubble, scalar triangle and scalar box integral functions. Deriving cut-constructible terms therefore reduces to the problem of finding the coefficients of these basis integrals. For the coefficients of scalar box integrals it was shown in [38] that a combination of *generalised* unitarity [19, 39, 40, 41], quadruple cuts in this case, along with

the use of *complex* momenta could be used, within a purely algebraic approach, to extract the desired coefficient from the cut integrand of the associated box topology.

Extracting triangle and bubble coefficients presents more of a problem. Unlike for the case of box coefficients, cutting all the propagators associated with the desired integral topology does not uniquely isolate a single integral coefficient. Inside a particular two-particle or triple cut lie multiple scalar integral coefficients corresponding to integrals with topologies sharing not only the same cuts but also additional propagators. These coefficients must therefore be disentangled in some way. There are multiple directions within the literature which have been taken to effect this separation. The pioneering work by Bern, Dixon, Dunbar and Kosower related unitarity cuts to Feynman diagrams and thence to the scalar integral basis, this then allowed for the derivation of many important results [17, 18, 19]. More recently the technique of Britto *et. al.* [23, 24, 25, 35, 36] has for two-particle cuts and the its extension to triple cuts by Mastrolia [42], highlighted the benefits of working in a spinor formalism, where the cut integrals can be integrated directly. Important results obtained in this way include the most difficult of the cut-constructable pieces for the one-loop amplitude for six gluons with the helicity configurations $A_6(+-+-+-)$ and $A_6(-+--++)$. The cut-constructible parts of Maximum-Helicity-Violating (MHV) one-loop amplitudes were found by joining MHV amplitudes together in a similar manner to at tree level [43]. This method has been applied by Bedford, Brandhuber, Spence and Travaglini to produce new QCD results [37]. In the approach of Ossola, Papadopoulos and Pittau [44, 45] it is possible to avoid the need to perform any integration or use any integral reduction techniques. Coefficients are instead extracted by solving sets of equations. The solutions of these equations include the desired coefficients, along with additional "spurious" terms corresponding to coefficients of terms which vanish after integrating over the loop momenta.

The many-fold different processes and their differing parton contents that will be needed at current and future collider experiments suggests that some form of automation, even of the more efficient "on-shell" techniques, will be required. From an efficiency standpoint, therefore, we would ideally wish to minimise the degree of calculation required for each step of any such process. Here we propose a new method for the extraction of scalar integral coefficients which aims to meet this goal. The technique follows in the spirit of the simplicity of the derivation of scalar box coefficients given in ref. [38]. Desired coefficients can be *constructed* directly using two-particle or triple cuts. The complete one-loop amplitude

can then be obtained by summing over all such cuts and adding any box terms and rational pieces. Alternatively our technique can be used to *extract* the bubble and triangle coefficients from a one-loop amplitude, generated for example from a Feynman diagram. Hence the technique is acting as an efficient way to perform the integration.

We use unitarity cuts to freeze some of the degrees of freedom of the integral loop momentum, whilst leaving others unconstrained. This then isolates a specific single bubble or triangle integral topology and hence its coefficient. Within each cut there remain additional coefficients. In the triangle case those of scalar box integrals. In the bubble case both scalar box and scalar triangle integrals contribute. Disentangling our desired coefficient from these extra contributions is a straightforward two step procedure. First one rewrites the loop momentum inside the cut integrand in terms of its unconstrained parameters. In the triangle case there is a single parameter, and in the bubble case there are a pair of parameters. Examining the behaviour of the integrand as these unconstrained parameters approach infinity then allows for a straightforward separation of the desired coefficient from any extra contributions. The coefficient of each basis integral function can therefore be extracted individually in an efficient manner with no further computation.

This paper is organised as follows. In section II we outline the notation used throughout this paper. In section III we proceed to present the basic structure of a one-loop amplitude in terms of a basis of scalar integral functions. We describe in section IV our procedure for extracting the coefficients of scalar triangle coefficients through the use of a particular loop-momentum parameterisation for the triple cuts along with the properties of the cut as the single free integral parameter tends to infinity. Section V extends this formalism to include the extraction of scalar bubble coefficients. The two-particle cut used in this case contains an additional free parameter and requires an additional step in our procedure. Finally in section VI we conclude by providing some applications which act as checks of our method. Initially we examine the extraction of various basis integral coefficients from some common one-loop integral functions. We then turn our attention to the construction of the coefficients of some more phenomenologically interesting processes. These include the three-mass triangle coefficient for the six photon amplitude $A_6(-+-+-+)$, as well as a representative three-mass triangle coefficient of the process $e^+e^- \to q^+\overline{q}^-g^-g^+$. Finally we construct the complete cut-containing part of the amplitude $A_6^{1-\mathrm{loop}}(1^-,2^-,3^+,4^+,5^+)$ and discuss further comparisons against coefficients of more complicated gluon amplitudes

contained in the literature.

II. NOTATION

In this section we summarise the notation used in the remainder of the paper. We will use the spinor helicity formalism [47, 48], in which the amplitudes are expressed in terms of spinor inner-products,

$$\langle j\, l\rangle = \langle j^-|l^+\rangle = \bar{u}_-(k_j)u_+(k_l)\,, \qquad\qquad [j\, l] = \langle j^+|l^-\rangle = \bar{u}_+(k_j)u_-(k_l)\,, \tag{2.1}$$

where $u_\pm(k)$ is a massless Weyl spinor with momentum k and positive or negative chirality. The notation used here follows the QCD literature, with $[i\, j] = \mathrm{sign}(k_i^0 k_j^0)\langle j\, i\rangle^*$ for real momenta so that,

$$\langle i\, j\rangle[j\, i] = 2k_i\cdot k_j = s_{ij}\,. \tag{2.2}$$

Our convention is that all legs are outgoing. We also define,

$$\lambda_i \equiv u_+(k_i), \qquad \tilde{\lambda}_i \equiv u_-(k_i)\,. \tag{2.3}$$

We denote the sums of cyclicly-consecutive external momenta by

$$K^\mu_{i\ldots j} \equiv k_i^\mu + k_{i+1}^\mu + \cdots + k_{j-1}^\mu + k_j^\mu\,, \tag{2.4}$$

where all indices are mod n for an n-gluon amplitude. The invariant mass of this vector is

$$s_{i\ldots j} \equiv K^2_{i\ldots j}\,. \tag{2.5}$$

Special cases include the two- and three-particle invariant masses, which are denoted by

$$s_{ij} \equiv K^2_{ij} \equiv (k_i + k_j)^2 = 2k_i\cdot k_j, \qquad\qquad s_{ijk} \equiv (k_i + k_j + k_k)^2\,. \tag{2.6}$$

We also define spinor strings,

$$\begin{aligned}\left\langle i^-\right|(\not{a} \pm \not{b})\left|j^-\right\rangle &= \langle i\, a\rangle[a\, j] \pm \langle i\, b\rangle[b\, j]\,,\\ \left\langle i^+\right|(\not{a} + \not{b})(\not{c} + \not{d})\left|j^-\right\rangle &= [i\, a]\left\langle a^-\right|(\not{c} + \not{d})\left|j^-\right\rangle + [i\, b]\left\langle b^-\right|(\not{c} + \not{d})\left|j^-\right\rangle\,.\end{aligned} \tag{2.7}$$

III. UNITARITY CUTTING TECHNIQUES AND THE ONE-LOOP INTEGRAL BASIS

Our starting point will be the general dimensionally-regularised decomposition of a one-loop amplitude into a basis of scalar integral functions [18, 53]

$$A_n^{1-\text{loop}} = \mathcal{R}_n + r_\Gamma \frac{(\mu^2)^\epsilon}{(4\pi)^{2-\epsilon}} \left(\sum_i b_i B_0(K_i^2) + \sum_{ij} c_{ij} C_0(K_i^2, K_j^2) + \sum_{ijk} d_{ijk} D_0(K_i^2, K_j^2, K_k^2) \right). \quad (3.1)$$

The scalar bubble, triangle and box integral functions are denoted by B_0, C_0 and D_0 respectively, and along with r_Γ their explicit forms can be found in Appendix C. The b_i, c_{ij} and d_{ijk} are their corresponding rational coefficients. Any ϵ dependence within these coefficients has been removed and placed into the rational, $\mathcal{R}_n$, term. The problem of deriving the one-loop amplitude is therefore reduced to that of finding the coefficients of these scalar integral functions and any rational terms when working in $D = 4$ dimensions.

We are going to consider obtaining these coefficients via the application of various cuts within the framework of generalised unitarity [19, 39, 40, 41]. In general our cut momenta will be complex, so for our purposes we define a "cut" as the replacement

$$\frac{i}{(l+K_i)^2} \rightarrow (2\pi)\delta((l+K_i)^2). \quad (3.2)$$

By systematically *constructing* all possible unitarity cuts we can reproduce every integral coefficient of a particular amplitude. Alternatively, application of the same procedure of "cutting" legs can be used to *extract* from a one-loop integral the corresponding coefficients of the standard basis integrals making up that particular integral, in a sense acting as a form of specialised integral reduction. This approach follows in a similar vein to that adopted by Ossola, Papadopoulos and Pittau [44].

The most straightforward implementation of the technique we present here is when the cut loop momentum is massless and kept in $D = 4$ dimensions. Eq. 3.1 therefore contains, within the term $\mathcal{R}_n$, any rational terms missed by performing cuts in only $D = 4$. Approaches for deriving such terms independently of unitarity cuts exist and so we do not concern ourselves with these here [23, 24, 26, 27, 29, 30, 31, 32, 33, 34, 44, 45].

As was demonstrated in [38], the application of a quadruple cut, as shown in figure 1, to $A_n^{1-\text{loop}}$ uniquely identifies a particular box integral topology $D_0(K_i^2, K_j^2, K_k^2)$ and hence its

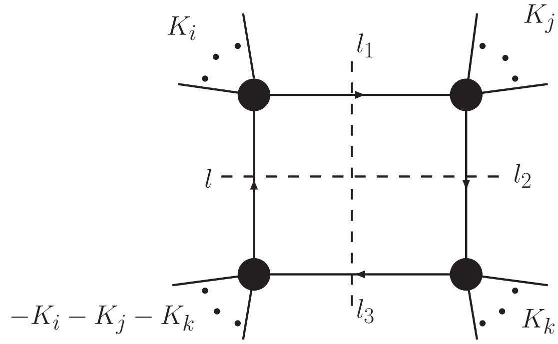

FIG. 1: A generic quadruple cut used to isolate the scalar box integral $D_0(K_i^2, K_j^2, K_k^2)$.

coefficient. This coefficient is then given by

$$d_{ijk} = \frac{1}{2}\sum_{a=1}^{2} A_1(l_{ijk;a})A_2(l_{ijk;a})A_3(l_{ijk;a})A_4(l_{ijk;a}), \tag{3.3}$$

where $l_{ijk;a}$ is the a^{th} solution of the cut loop momentum l that isolates the scalar box function $D_0(K_i^2, K_j^2, K_k^2)$, there are 2 such solutions. Eq. 3.3 applies as well to the cases when one or more of the four legs of the box is massless. This is a result of the existence, for complex momenta, of a well-defined three-point tree amplitude corresponding to any corner of a box containing a massless leg.

Applying a triple cut to the amplitude $A_n^{1-\text{loop}}$ does not isolate a single basis integral. Instead we have a triangle integral plus a sum of box integrals obtained by "opening" a fourth propagator. This can be represented schematically via

$$r_\Gamma \frac{(\mu^2)^\epsilon}{(4\pi)^{2-\epsilon}} \left(c_{ij} C_0(K_i^2, K_j^2) + \sum_k d_{ijk} D_0(K_i^2, K_j^2, K_k^2) + \ldots \right), \tag{3.4}$$

where the additional terms correspond to "opening" the K_i leg or the K_j leg instead of the $-(K_i + K_j)$ leg. Similarly in the case of a two-particle cut we again cannot isolate a single basis integral by itself. Instead we get additional triangle and box integrals corresponding to "opening" third and forth propagators. Schematically this is given by

$$r_\Gamma \frac{(\mu^2)^\epsilon}{(4\pi)^{2-\epsilon}} \left(b_i B_0(K_i^2) + \sum_j c_{ij} C_0(K_i^2, K_j^2) + \sum_{jk} d_{ijk} D_0(K_i^2, K_j^2, K_k^2) + \ldots \right), \tag{3.5}$$

where again the additional terms are boxes with the K_i leg or the K_j legs "opened". Whilst not isolating a single integral each of the above cuts does single out either *one* scalar triangle, in the triple cut case, or *one* scalar bubble, in the two-particle cut case. Disentangling

these single bubble or triangle integral functions from the contributions of the remaining basis integrals will allow us to directly read off the corresponding coefficient. Applying all possible two-particle, triple and quadruple cuts then enables us to derive the coefficients of every basis integral function.

IV. TRIPLE CUTS AND SCALAR TRIANGLE COEFFICIENTS

A triple cut contains not only contributions for the corresponding scalar triangle integral, but also contributions from scalar box integrals which share the same three cuts as the triangle. Of the four propagators of a scalar box integral, three will be given by the three cut legs of the triple cut loop integral. The forth propagator will be contained inside the cut integrand in a denominator factor of the form $(l-P)^2$, which corresponds to a propagator pole. Ideally we want to separate terms containing such poles from the remainder of the cut integrand. The remaining term will be the scalar triangle integral multiplied by its coefficient for that particular cut.

The three delta functions of a triple cut constrain the cut loop momentum such that only a single free parameter of the integral remains, which we label t. We can express the loop momentum in terms of this parameter using the orthogonal null four-vectors, a_i^μ, with $i = 1, 2, 3$, specific forms for these basis vectors are presented in section IV A. The loop momentum is then given by

$$l^\mu = a_0^\mu t + \frac{1}{t} a_1^\mu + a_2^\mu. \tag{4.1}$$

Denominator factors of the cut integrand depending upon the cut loop momentum, can be written as propagators of the general form, $(l-P)^2$. When these propagators go on-shell they will correspond to poles in t. These poles will be solutions of the following equation

$$(l-P)^2 = 0 \quad \Rightarrow \quad 2(a_0 \cdot P)t + 2(a_1 \cdot P)\frac{1}{t} + 2(a_2 \cdot P) - P^2 = 0. \tag{4.2}$$

If we consider t to be a complex parameter then we can use a partial fraction decomposition in terms of t to rewrite an arbitrary triple-cut integral. For the extraction of integral coefficients we need only work with integrals in $D = 4$ dimensions. We also drop an overall denominator factor of $1/(2\pi)^4$ which multiplies all integrals. The partial fraction decomposition is therefore given, in the case when we have applied a triple cut on the legs l^2, $(l-K_1)^2$

and $(l-K_2)^2$, by

$$
\begin{aligned}
&(2\pi)^3\int d^4l\prod_{i=0}^{2}\delta(l_i^2)A_1A_2A_3\\
&\qquad=(2\pi)^3\int d^4l\prod_{i=0}^{2}\delta(l_i^2)\left(\left[\mathrm{Inf}_tA_1A_2A_3\right](t)+\sum_{\text{poles }\{j\}}\frac{\mathrm{Res}_{t=t_j}\,A_1A_2A_3}{t-t_j}\right),
\end{aligned}
\tag{4.3}
$$

where $l_i = l - K_i$ and $l_0 = l$. This is a sum of all possible poles of t, labelled here as the set $\{j\}$, contained in the cut integrand denoted by $A_1A_2A_3$. Pieces of the integrand without a pole are contained in the Inf term, originally given in [30], and defined such that

$$
\lim_{t\to\infty}\left(\left[\mathrm{Inf}_tA_1A_2A_3\right](t)-A_1(t)A_2(t)A_3(t)\right)=0. \tag{4.4}
$$

In general $[\mathrm{Inf}_zA_1A_2A_3](t)$ will be some polynomial in t,

$$
\left[\mathrm{Inf}_tA_1A_2A_3\right](t)=\sum_{i=0}^{m}f_it^i, \tag{4.5}
$$

where m is the leading degree of large t behaviour and depends upon the specific integrand in question.

After applying the three delta functions constraints we see that taking the residue of $A_1A_2A_3$ at a particular pole, $t = t_0$, removes any remaining dependence upon the loop momentum. Hence we can write

$$
\int d^4l\prod_{i=0}^{2}\delta(l_i^2)\frac{\mathrm{Res}_{t=t_0}\,A_1A_2A_3}{t-t_0}\sim\lim_{t\to t_0}\left[(t-t_0)A_1A_2A_3\right]\int d^4l\prod_{i=0}^{2}\delta(l_i^2)\frac{1}{t-t_0}. \tag{4.6}
$$

Where on the right hand side of this we understand the integral, $\int d^4l$, as over the parameterised form of l in terms of t and the three other degrees of freedom. In the cut integrand the only source of poles in t is from propagator terms of the type $1/(l-P)^2$. Generally each such propagator, when on-shell, contains two poles due to the quadratic nature, in t, of eq. (4.2). If we label these solutions $t_\pm$ then we can write a triple-cut scalar box in terms of these poles as

$$
\int d^4l\prod_{i=0}^{2}\delta(l_i^2)\frac{1}{(l-P)^2}\sim\frac{1}{t_+-t_-}\left(\int d^4l\prod_{i=0}^{2}\delta(l_i^2)\frac{1}{t-t_+}-\int d^4l\prod_{i=0}^{2}\delta(l_i^2)\frac{1}{t-t_-}\right). \tag{4.7}
$$

From comparing this to eq. (4.6) we see that all residue terms of eq. (4.3) simply correspond to pieces of triple-cut scalar box functions multiplied by various coefficients.

Therefore we can associate all residue terms with scalar boxes, meaning that our triple cut amplitude can be written simply as

$$(2\pi)^3 \int d^4 l \prod_{i=0}^{2} \delta(l_i^2) A_1 A_2 A_3 = (2\pi)^3 \int dt J_t \left(\sum_{i=0}^{m} f_i t^i \right) + \sum_{\text{boxes } \{l\}} d_l D_0^{\text{cut}}. \tag{4.8}$$

This is a sum over the set $\{l\}$ of possible cut scalar boxes, D_0^{cut}, and their associated coefficients, d_l, along with a power series in positive powers of t. In eq. (4.8) we have integrated over the three delta functions after performing the integral transformation from l^μ to t, the Jacobian of which, and any additional factors picked up from the integration is contained in the factor J_t. The limit m of the summation is the maximum power of t appearing in the integrand, which in turn is the maximum power of l appearing in the numerator of the integrand. In general for renormalisable theories, such as QCD amplitudes, $m \leq 3$.

We must now turn our attention to answering the question of what do the remaining terms correspond to? To do this we need to understand the behaviour of the integrals over positive powers of t. There is a freedom in our choice of the parameterisation of the cut-loop momentum. This freedom extends, as we will prove in section IV A, to choosing a parametrisation where the integrals over all positive powers of t vanish. Doing this then reduces the cut integrand to

$$(2\pi)^3 \int d^4 l \prod_{i=0}^{2} \delta(l_i^2) A_1 A_2 A_3 = (2\pi)^3 f_0 \int dt J_t + \sum_{\text{boxes } \{l\}} d_l D_0^{\text{cut}}. \tag{4.9}$$

The remaining integral is now simply that of a triple-cut scalar triangle, multiplied by the coefficient f_0. For the triple-cut scalar triangle integral, $C_0^{\text{cut}}(K_i^2, K_j^2)$, given by $-(2\pi)^3 \int dt J_t$, the triple cut form of eq. (C4), we find that its corresponding coefficient is given simply by

$$c_{ij} = - \left[\text{Inf}_t A_1 A_2 A_3 \right] (t) \Big|_{t=0}, \tag{4.10}$$

which is just the first term in the series expansion in t of the cut-integrand at infinity.

The simplicity of this result relies crucially upon two facts. The first is that on the triple cut the integral is sufficiently simple that it can be decomposed into either a triangle contribution or a box contribution. This is important as it allows us to easily distinguish between the two types of term. As an example consider a linear box which contains a numerator factor constructed such that it vanishes at the pole contained in the denominator,

but without being proportional to the denominator itself. To which basis integral does this term contribute to? In the simplest case such a term would look like

$$\int d^4l \prod_{i=0}^{2} \delta(l_i^2) \frac{\langle lW \rangle}{\langle lP \rangle} = \int d^4l \prod_{i=0}^{2} \delta(l_i^2) \frac{\langle aW \rangle (t - t_0)}{\langle aP \rangle (t - t_0)} = \frac{\langle aW \rangle}{\langle aP \rangle} \int d^4l \prod_{i=0}^{2} \delta(l_i^2),$$

and hence must contribute entirely to the triangle integral, it contains no box terms. Here we have chosen a simplified loop momentum parameterisation in terms of two basis spinors $|a^+\rangle$ and $|\overline{a}^+\rangle$ such that $\langle lP \rangle = t\langle aP \rangle + \langle \overline{a}P \rangle$. This then contains a pole in t at $t_0 = -\langle \overline{a}P \rangle / \langle aP \rangle$ and we have chosen the spinor $|W^+\rangle$ such that $\langle \overline{a}W \rangle = -t_0 \langle aW \rangle$.

The second crucial fact is the *vanishing* of the other integrals over t so that the complete scalar triangle integral is given by only the remaining integral over t^0. Hence the coefficient is given by a single term. Furthermore, the use of a complex loop momentum also means that we can apply this formalism to the extraction of scalar coefficients corresponding to one- and two-mass triangles as well as three-mass triangles. As discussed above for the case of box coefficients, this is a result of the possibility of a well-defined three-point vertex when using complex momentum, enabling in these cases the construction of non-vanishing cut integrands.

A. The momentum parameterisation

We wish to compute the coefficient of the scalar triangle singled out by the triple cut given in figure 2. The cut integral when written in terms of tree amplitudes is

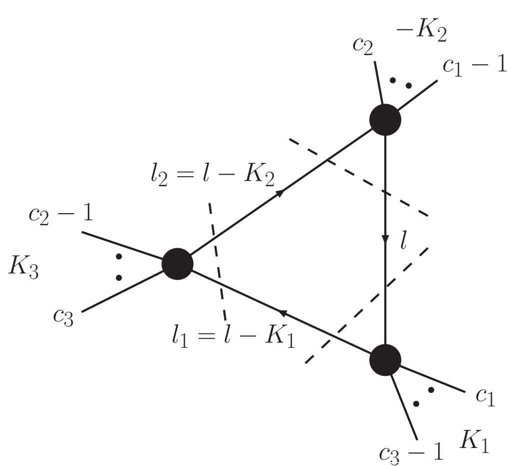

FIG. 2: The triple cut used to compute the scalar triangle coefficient of $C_0(K_1^2, K_2^2)$.

$$(2\pi)^3 \int d^4l \prod_{i=0}^{2} \delta(l_i^2) A^{\rm tree}_{c_3-c_1+2}(-l, c_1, \ldots, (c_3-1), l_1) A^{\rm tree}_{c_2-c_3+2}(-l_1, c_3, \ldots, (c_2-1), l_2)$$
$$\times A^{\rm tree}_{n-c_2+c_1+2}(-l_2, c_2, \ldots, (c_1-1), l), \tag{4.11}$$

with $l_1 = l - K_1 = l - K_{c_1\ldots c_3-1}$ and $l_2 = l - K_2 = l + K_{c_2\ldots c_1-1}$, so that $K_1 = K_{c_1\ldots c_3-1}$ and $K_2 = -K_{c_2\ldots c_1-1}$.

Our first step will be to find a parameterisation of l in terms of the single free integral parameter remaining after satisfying all three of the cut delta functions constraints,

$$l^2 = 0, \qquad l_1^2 = (l-K_1)^2 = 0, \quad \text{and} \quad l_2^2 = (l-K_2)^2 = 0. \tag{4.12}$$

Each of the three legs can be massive or massless. We will deal with the general case of three massive legs explicitly here. The cases with massless legs are then easily found by setting the relevant mass in the parameterisation to zero. We will find it very convenient to express l^μ in terms of a basis of momentum identical to the momenta l_1 and l_2 used by Ossola, Papadopoulos and Pittau [44]. We will write these momenta in the suggestive notation $K_1^\flat$ and $K_2^\flat$ and define them via

$$\begin{aligned} K_1^{\flat,\mu} &= K_1^\mu - \frac{S_1}{\gamma} K_2^{\flat,\mu}, \\ K_2^{\flat,\mu} &= K_2^\mu - \frac{S_2}{\gamma} K_1^{\flat,\mu}, \end{aligned} \tag{4.13}$$

with $\gamma = \langle K_1^{\flat,-} | \not{K}_2^\flat | K_1^{\flat,-} \rangle \equiv \langle K_2^{\flat,-} | \not{K}_1^\flat | K_2^{\flat,-} \rangle$ and $S_i = K_i^2$. Each momentum $K_1^\flat$, $K_2^\flat$ is the massless projection of one of the massive legs in the direction of the other masslessly projected leg. A more practical definition of $K_1^\flat$ and $K_2^\flat$, in terms of the external momenta alone, can be found by solving the above equations for $K_1^\flat$ and $K_2^\flat$, so that in terms of S_1, S_2, K_1^μ and K_2^μ we have

$$K_1^{\flat,\mu} = \frac{K_1^\mu - (S_1/\gamma) K_2^\mu}{1 - (S_1 S_2/\gamma^2)}, \qquad K_2^{\flat,\mu} = \frac{K_2^\mu - (S_2/\gamma) K_1^\mu}{1 - (S_1 S_2/\gamma^2)}. \tag{4.14}$$

In addition γ can be expressed in terms of the external momenta,

$$\gamma_\pm = (K_1 \cdot K_2) \pm \sqrt{\Delta}, \qquad \Delta = (K_1 \cdot K_2)^2 - K_1^2 K_2^2. \tag{4.15}$$

When using eq. (4.10) we must average over the number of solutions of γ. In the three-mass case there are a pair of solutions. For the one- and two-mass cases, when either $K_1^2 = 0$ or $K_2^2 = 0$, then there is only a single solution.

After satisfying the three constraints given by eq. (4.12) we write the spinor components of l^μ in terms of our basis $K_1^\flat$ and $K_2^\flat$ as

$$\begin{aligned} \langle l^-| &= t\langle K_1^{\flat,-}| + \alpha_{01}\langle K_2^{\flat,-}|, \\ \langle l^+| &= \frac{\alpha_{02}}{t}\langle K_1^{\flat,+}| + \langle K_2^{\flat,+}|, \end{aligned} \tag{4.16}$$

where

$$\alpha_{01} = \frac{S_1\,(\gamma - S_2)}{(\gamma^2 - S_1S_2)}, \qquad \alpha_{02} = \frac{S_2\,(\gamma - S_1)}{(\gamma^2 - S_1S_2)}. \tag{4.17}$$

Written as a four-vector, l^μ is given by

$$l^\mu = \alpha_{02}K_1^{\flat,\mu} + \alpha_{01}K_2^{\flat,\mu} + \frac{t}{2}\langle K_1^{\flat,-}|\gamma^\mu|K_2^{\flat,-}\rangle + \frac{\alpha_{01}\alpha_{02}}{2t}\langle K_2^{\flat,-}|\gamma^\mu|K_1^{\flat,-}\rangle. \tag{4.18}$$

We can also use momentum conservation to write component forms for the other two cut momenta l_i with $i = 1, 2$,

$$\begin{aligned} \langle l_i^-| &= t\langle K_1^{\flat,-}| + \alpha_{i1}\langle K_2^{\flat,-}|, \\ \langle l_i^+| &= \frac{\alpha_{i2}}{t}\langle K_1^{\flat,+}| + \langle K_2^{\flat,+}|, \end{aligned} \tag{4.19}$$

where the α_{ij} are given in Appendix A.

A final point is that after having integrated over the three delta function constraints and performed the change of variables to the momentum parameterisation of eq. (4.16) we have the factor $J_t = 1/(t\gamma)$ contained in eq. (4.8). We always associate this factor with the scalar triangle integral and so its explicit form does not play a role in our formalism.

B. Vanishing integrals

As we have remarked previously, the simplicity of the method outlined here rests crucially upon the properties of the momentum parameterisation we have used. The key feature is the *vanishing* of the integrals over t. It can easily be shown that within our chosen momentum parameterisation, of section IV A, any integral of a positive or negative power of t vanishes. Following an argument very similar to that used by Ossola, Papadopoulos and Pittau [44] we use $\langle K_1^{\flat,\pm}|\not{K}_1|K_2^{\flat,\pm}\rangle = 0$, $\langle K_1^{\flat,\pm}|\not{K}_2|K_2^{\flat,\pm}\rangle = 0$ and $\langle K_1^{\flat,\pm}|\gamma^\mu|K_2^{\flat,\pm}\rangle\langle K_1^{\flat,\pm}|\gamma_\mu|K_2^{\flat,\pm}\rangle = 0$, to show that

$$\begin{aligned} \int d^4l\frac{\langle K_1^{\flat,-}|\not{l}|K_2^{\flat,-}\rangle^n}{l^2l_1^2l_2^2} = 0 \quad &\Rightarrow \quad \int dtJ_t\frac{1}{t^n} = 0 \quad \text{for } n \geq 1, \\ \int d^4l\frac{\langle K_2^{\flat,-}|\not{l}|K_1^{\flat,-}\rangle^n}{l^2l_1^2l_2^2} = 0 \quad &\Rightarrow \quad \int dtJ_tt^n = 0 \quad \text{for } n \geq 1. \end{aligned} \tag{4.20}$$

The vanishing of these terms then leads directly to our general procedure, encapsulated in eq. (4.10), which is to simply express the triple cut of the desired scalar triangle in the momentum parameterisation given by eq. (4.16) and then take the t^0 component of a series expansion in t around infinity.

V. TWO-PARTICLE CUTS AND SCALAR BUBBLE COEFFICIENTS

In the same spirit as the triangle case we now wish to extract the coefficients of scalar bubble terms using, in this case, a two-particle cut. Now a two-particle cut will contain in addition to our desired scalar bubble both scalar boxes and triangles, all of which need to be disentangled. What we will find, though, is that naively applying the technique as given for the scalar triangle coefficients will not give us the complete scalar bubble contribution.

The reason for this is straightforward to see. A two-particle cut places only two constraints on the loop momentum and so we can parameterise it in terms of two free variables, which we will label t and y. Consider rewriting the cut integrand in a partial fraction decomposition in terms of y. Schematically, therefore, the two-particle cut of the legs l^2 and $(l-K_1)^2$ can be written as

$$(2\pi)^2\int d^4l \prod_{i=0}^{1}\delta(l_i^2)A_1A_2 = (2\pi)^2\int dtdy J_{t,y}\left(\left[\mathrm{Inf}_y A_1A_2\right](y) + \sum_{\mathrm{poles}\ \{j\}} \frac{\mathrm{Res}_{y=y_j} A_1A_2}{y-y_j}\right), \quad (5.1)$$

where again $\{j\}$ is the sum over all possible poles, this time in y, and $J_{t,y}$ contains any terms from the change into the parameterisation of y and t as well as any pieces picked up by integrating over the two delta functions. So far this seems to be similar to the triangle case, but with the residue terms now corresponding to triangles as well as boxes. As we have two parameters though we can consider a further partial fraction decomposition, this time with t, giving

$$\begin{aligned}(2\pi)^2\int d^4l \prod_{i=0}^{1}\delta(l_i^2)A_1A_2 = \\ (2\pi)^2\int dtdy J_{t,y}\Bigg(&\left[\mathrm{Inf}_t\left[\mathrm{Inf}_y A_1A_2\right](y)\right](t) + \left[\mathrm{Inf}_t\left(\sum_{\mathrm{poles}\ \{j\}}\frac{\mathrm{Res}_{y=y_j} A_1A_2}{y-y_j}\right)\right](t) \\ &+ \sum_{\mathrm{poles}\ \{l\}}\frac{\mathrm{Res}_{t=t_l}\left[\mathrm{Inf}_y A_1A_2\right](y)}{t-t_l} + \sum_{\mathrm{poles}\ \{j\},\{l\}}\frac{\mathrm{Res}_{t=t_l}\left[\frac{\mathrm{Res}_{y=y_j} A_1A_2}{y-y_j}\right]}{t-t_l}\Bigg), \quad (5.2)\end{aligned}$$

where here $\{l\}$ is the sum over all possible poles in t. The general dependence of the cut integral momentum, l^μ, on the free integral parameters t and y can be written in terms of null four-vectors a_i^μ with $i = 0, 1, 2, 3, 4$ such that $l^2 = 0$. An explicit form for these will be presented in section V A. We then define l^μ by

$$l^\mu = \frac{y^2}{t} a_0^\mu + \frac{y}{t} a_1^\mu + y a_2^\mu + t a_3^\mu + a_4^\mu. \tag{5.3}$$

Again residues of pole terms will correspond to the solutions of $(l - P)^2 = 0$ and hence it is straightforward to see that the final term of eq. (5.2), containing the sum of residues in both y and t, has both of these free parameters fixed. Any such terms must contain at least one propagator pole. Also the numerator will be independent of any integration variables, as both y and t are fixed. Thus all such terms will correspond to purely scalar triangle and scalar box terms. Looking at the second and third terms of eq. (5.2) we might also, at least initially, want to associate these terms with contributions to scalar triangle terms only and hence naively conclude that only the first term of eq. (5.2) contributes to the scalar bubble coefficient. This assumption though would be wrong.

The crucial difference between the single residue terms of eq. (5.2) and those of eq. (4.3) is the parameterisation of the loop momentum which is being used. Taking the residue of a pole term at a particular point y freezes y such that we force a particular momentum parameterisation upon these triple-cut terms. Importantly, in general this particular forced momentum parameterisation is such that the integrals over t in the second and third terms of eq. (5.2) now no longer vanish.

If only scalar triangle contributions came from the integrals over t then this would not be an issue; we could just discard these terms as not relevant for the extraction of our bubble coefficient. What we find though, through a simple application of Passarino-Veltman reduction techniques, is that these integrals contain scalar bubble contributions, B_0, with coefficients b,

$$\int dt J_t' t^n = b\, B_0 + c\, C_0, \tag{5.4}$$

where J_t' is the relevant Jacobian for this parameterisation of the loop momentum and c is the coefficient corresponding to the scalar triangle contribution, C_0. We cannot therefore simply discard the residue pieces of eq. (5.2), as we could in the triangle case, if we want to derive the full scalar bubble coefficient. Furthermore, there is an additional complication.

We will see that the integrals over powers of y contained in the first term of eq. (5.2) also do not vanish in general and hence must also be taken into account.

There is a limit to the maximum positive powers of y and t that appear in the rewritten partial-fractioned decomposition of the integral. For renormalisable theories, such as QCD, up to three powers of t appear for triangle coefficients and up to four powers of y for bubble coefficients. Therefore the power series in y and t of the Inf operators will always terminate at these fixed points. It is then straightforward, as we will discuss in section V D and section V B, to derive the general form for all possible non-vanishing contributing integrals, over powers of y and t, in terms of their scalar bubble contributions.

Calculation of the scalar bubble coefficient therefore requires a two stage process. First take the Inf_y and Inf_t pieces of the cut integrand and replace any integrals over y with their known *general* forms, as we shall see integrals proportional to t will vanish. Secondly compute all possible triple cuts that could be generated by applying a third cut to the two-particle cut we are considering. To these terms then apply, not the parameterisation we used in section IV, but the parameterisation forced upon us by taking the residues of the poles in y, which we will derive in section V C. This is equivalent to calculating all the contributions from the residues of the partial fraction decomposed cut integrand of eq. (5.2). Within these terms we then replace any integrals of powers of t with their known *general* forms. Finally we sum all the contributing pieces together to get the full scalar bubble contribution and hence its coefficient. Our final result for assembling the bubble coefficient is then given by eq. (5.28).

A. The momentum parameterisation for the two-particle cut

We want to extract the scalar bubble coefficient obtainable from the application of the two-particle cut given in figure 3. This two-particle cut can be expressed in terms of tree amplitudes as

$$(2\pi)^2\int d^4l \prod_{i=0}^{1}\delta(l_i^2)A^{\text{tree}}_{c_2-c_1+2}(-l,(c_1+1),\ldots,c_2,l_1)A^{\text{tree}}_{n-c_2+c_1+2}(-l_1,(c_2+1),\ldots,c_1,l),\qquad(5.5)$$

with $l_1 = l - K_{c_1+1\ldots c_2} = l - K_1$.

A bubble can be classified entirely in terms of the momentum of one of its two legs, which we label K_1, and so we will find it useful to express the cut loop momentum l in terms of

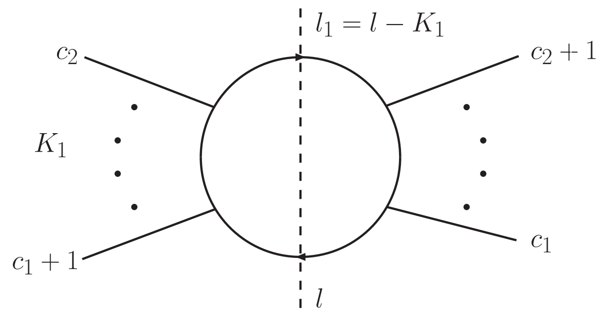

FIG. 3: The two-particle cut for computing the scalar bubble coefficient of $B_0(K_1^2)$.

the pair of massless momenta $K_1^\flat$ and χ defined via

$$K_1^{\flat,\mu} = K_1^\mu - \frac{S_1}{\gamma}\chi^\mu, \tag{5.6}$$

here $\gamma = \langle\chi^\pm|\not{K}_1|\chi^\pm\rangle \equiv \langle\chi^\pm|\not{K}_1^\flat|\chi^\pm\rangle$. The arbitrary vector χ can be chosen independently for each bubble coefficient as a result of the independence of the choice of basis representation for the cut momentum. In the two-particle cut case we have only two momentum constraints

$$l^2 = 0, \quad \text{and} \quad l_1^2 = (l - K_1)^2 = 0, \tag{5.7}$$

and so we have two free parameters which we will label y and t. The loop momentum can then be expressed in terms of spinor components as

$$\begin{aligned} \langle l^-| &= t\langle K_1^{\flat,-}| + \frac{S_1}{\gamma}(1-y)\langle\chi^-|, \\ \langle l^+| &= \frac{y}{t}\langle K_1^{\flat,+}| + \langle\chi^+|. \end{aligned} \tag{5.8}$$

Written as a four-vector l^μ is

$$l^\mu = yK_1^{\flat,\mu} + \frac{S_1}{\gamma}(1-y)\chi^\mu + \frac{t}{2}\langle K_1^{\flat,-}|\gamma^\mu|\chi^-\rangle + \frac{S_1}{2\gamma}\frac{y}{t}(1-y)\langle\chi^-|\gamma^\mu|K_1^{\flat,-}\rangle. \tag{5.9}$$

We can also use momentum conservation to write a component form for the other cut momentum. We have

$$\begin{aligned} \langle l_1^-| &= \langle K_1^{\flat,-}| - \frac{S_1}{\gamma}\frac{y}{t}\langle\chi^-|, \\ \langle l_1^+| &= (y-1)\langle K_1^{\flat,+}| + t\langle\chi^+|. \end{aligned} \tag{5.10}$$

Furthermore after rewriting the integral in this cut-momentum parameterisation and integrating over the two delta function constraints we find the following simple result for the constant $J_{t,y}$ contained in eq. (5.1), namely $J_{t,y} = 1$.

B. Non-vanishing integrals

In the case of the scalar triangles of section IV B crucial simplifications occurred as a result of our chosen cut momentum parameterisation. Any integral over a power of t vanished, leaving only a single contribution corresponding to the desired coefficient. For the scalar bubble coefficient things are not quite as simple.

We can use $\langle K_1^{\flat,\pm}|\not{K}_1|\chi^\pm\rangle = 0$ as well as $\langle K_1^{\flat,\pm}|\gamma^\mu|\chi^\pm\rangle\langle K_1^{\flat,\pm}|\gamma_\mu|\chi^\pm\rangle = 0$ to show that

$$\begin{aligned}
\int d^4l \frac{\langle\chi^-|\not{l}|K_1^{\flat,-}\rangle^n}{l^2 l_1^2} &= 0 \quad \Rightarrow \quad \int dtdy\ t^n = 0, \\
\int d^4l \frac{\langle K_1^{\flat,-}|\not{l}|\chi^-\rangle^n}{l^2 l_1^2} &= 0 \quad \Rightarrow \quad \int dtdy \left(\frac{y}{t}\right)^n (1-y)^n = 0.
\end{aligned} \tag{5.11}$$

Hence the integrals over all positive and negative powers of t vanish,

$$\int dtdy\ t^n = 0 \quad \text{for } n \neq 0. \tag{5.12}$$

Integrals over positive powers of y, contained within the double Inf piece of the first term of eq. (5.2), will *not* vanish. These integrals are straightforwardly derivable with the aid of identities involving the four vector $n^\mu = K_1^{\flat,\mu} - (S_1/\gamma)\chi^\mu$ which satisfies the constraints $(K_1 \cdot n) = 0$ and $n^2 = -S_1$. It is then possible to show the following relations in $D = 4$ dimensions, and remembering that $J_{t,y} = 1$,

$$\begin{aligned}
\int d^4l \frac{(l\cdot n)^{2m-1}}{l^2 l_1^2} = 0 \quad &\Rightarrow \quad \int dtdy \left(\frac{1}{2} - y\right)^{2m-1} = 0, \\
\int d^4l \frac{(l\cdot n)^{2m}}{l^2 l_1^2} = S_1^{2m} B_{\mathcal{PV}}^m \quad &\Rightarrow \quad \int dtdy \left(\frac{1}{2} - y\right)^{2m} = S_1^{2m} \tilde{B}_{\mathcal{PV}}^m, \\
\int d^4l \frac{(l\cdot K_1)^{2m}}{l^2 l_1^2} = (2m+1) S_1^{2m} B_{\mathcal{PV}}^m \quad &\Rightarrow \quad \frac{1}{2^{2m}} \int dtdy = (2m+1) S_1^{2m} \tilde{B}_{\mathcal{PV}}^m,
\end{aligned} \tag{5.13}$$

where $B_{\mathcal{PV}}^m$ and $\tilde{B}_{\mathcal{PV}}^m$ are Passarino-Veltman reduction coefficients, the explicit forms of which are not needed. Solving these equations for the integral of y^m leads to the result

$$\int dtdy\ y^m = \frac{1}{m+1} \int dtdy \quad \text{for } m \geq 0. \tag{5.14}$$

Contributions to our desired scalar bubble coefficient from the double Inf piece of eq. (5.2) therefore come not only from the single constant $t^0 y^0$ term but also from terms proportional to integrals of $t^0 y^m$. This is not the end of the story. As described above, there can be further contributions from the second and third residue terms generated in the decomposition of

eq. (5.2). We could proceed from the cut integrand to explicitly calculate these residue terms. However as we will shall see, a more straightforward approach is to derive these terms by relating them to triple cuts.

C. The momentum parameterisation for triple cut contributions

We wish to relate the contributions to the bubble coefficient of the residue pieces, separated in the decomposition of eq. (5.2), to triple cuts in a specific basis of the cut-loop momentum. To find this basis we will apply the additional constraint

$$(l+K_2)^2=0, \tag{5.15}$$

to the two-particle cut momentum of section V A. Note that here we label the "K_2" leg as K_2 in contrast to $(-K_2)$ as we did in the triangle coefficient case of section IV A. This constraint corresponds to the application of an additional cut which would appear as $\delta((l+K_2)^2)$ inside the integral. This additional constraint, applied to the starting point of the two-particle cut loop momentum, forces us to use $K_1^\flat$ and χ as the momentum basis vectors of l. Importantly, this differs from the basis choice for the triple cut momenta developed in section IV A, which leads to the differing behaviour of these triple-cut contributions.

The presence of y in both $\langle l^-|$ and $\langle l^+|$ directs us for reasons of efficiency to choose to use eq. (5.15) to first constrain y, leaving t free. Looking at eq. (5.9) we see that as l^μ is quadratic in y then there are two solutions to this constraint, $y_\pm$, which are given by

$$\begin{aligned} y_\pm = {}& \frac{1}{2S_1\langle\chi^-|\not{K}_2|K_1^{\flat,-}\rangle}\Bigg(\Big(\gamma\langle K_1^{\flat,-}|\not{K}_2|K_1^{\flat,-}\rangle - S_1\langle\chi^-|\not{K}_2|\chi^-\rangle\Big)\,t + S_1\langle\chi^-|\not{K}_2|K_1^{\flat,-}\rangle \\ & \pm\sqrt{\Big(S_1\langle\chi^-|\not{K}_2|K_1^{\flat,-}\rangle + 2t\gamma\,(K_1\cdot K_2)\Big)^2 - 4S_1S_2\gamma t\left(t\gamma - \langle\chi^-|\not{K}_2|K_1^{\flat,-}\rangle\right)}\Bigg). \end{aligned} \tag{5.16}$$

On substituting these two solutions into the two-particle cut momentum of eq. (5.8) we obtain our desired triple-cut momentum parameterisation.

Our final step is then to relate the triple-cut integrals defined in this basis to the residue terms of eq. (5.2). Rewriting the triple cut integral after the change of momentum parameterisation and integrating over all but the third delta function gives the general form

$$(2\pi)^3\int dt dy\; J_t'\left(\delta(y-y_+)+\delta(y-y_-)\right)\mathcal{M}(y,t), \tag{5.17}$$

where $\mathcal{M}(y,t)$ is a general cut integrand and

$$J'_t = \frac{1}{\sqrt{\left(S_1\langle\chi^-|\not{K}_2|K_1^{\flat,-}\rangle + 2t\gamma\,(K_1\cdot K_2)\right)^2 - 4S_1S_2\gamma t\left(t\gamma - \langle\chi^-|\not{K}_2|K_1^{\flat,-}\rangle\right)}}. \tag{5.18}$$

Upon examination of a general residue term we find that it corresponds to an integral of the form

$$(2\pi)^2 i\int dtdy J_{t,y}\operatorname*{Res}_{y=y_\pm}\frac{\mathcal{M}(y,t)}{(l+K_2)^2} \equiv -\frac{(2\pi)^3}{2}\int dtdy\; J'_t\left(\delta(y-y_+)+\delta(y-y_-)\right)\mathcal{M}(y,t), \tag{5.19}$$

and hence that residue contributions are given, up to a factor of $(-1/2)$, by the triple cut.

This result applies equally when $S_2 = K_2^2 = 0$, corresponding to a one or two-mass triangle, when the appropriate scale is set to zero in eq. (5.16) and eq. (5.18). The momentum parameterisation in this simplified case is contained in Appendix B.

D. More non-vanishing integrals and bubble coefficients

There is a direct correspondence between a triple cut contribution and a residue contribution. The sum of all possible triple cuts, which contain the original two-particle cut, will therefore correspond to the sum of all residue terms. We must now examine how such terms contribute to the bubble coefficient itself.

Unlike for the case of triple cut integrands as parameterised in section IV A we will find that there are contributions, specifically in this case bubble coefficient contributions, coming from the integrals over t. To see this let us investigate the integrals over t in more detail. As an example consider extracting the scalar bubble term coming from a two-mass linear triangle (with the massless leg K_2 so that $S_2 = 0$). We would start from a two-particle cut which, after decomposing as eq. (5.1), would give

$$(2\pi)^2\int d^4l\prod_{i=0}^{1}\delta(l_i^2)\frac{\langle K_2^-|\not{l}|a^-\rangle}{(l+K_2)^2} \tag{5.20}$$

$$= (2\pi)^2\int d^4l\prod_{i=0}^{1}\delta(l_i^2)\frac{[la]}{[lK_2]} = (2\pi)^2\int dtdyJ'_t\left(\frac{[K_1^\flat a]}{[K_1^\flat K_2]} + \frac{\operatorname{Res}_{y=-t\frac{[\chi K_2]}{[K_1^\flat K_2]}}\left(\frac{y[K_1^\flat a]+t[\chi a]}{y+t\frac{[\chi K_2]}{[K_1^\flat K_2]}}\right)}{[K_1^\flat K_2]\left(\frac{y}{t}[K_1^\flat K_2]+[\chi K_2]\right)}\right).$$

The first term of this is clearly not the complete coefficient, and so we need to obtain the bubble contribution contained within the second term. Consider reconstructing this term

using a triple cut with the cut loop momentum parameterised in a form given by setting y equal to its value at the residue of the pole of this second term. This triple cut term is given by

$$
\begin{aligned}
&-\frac{(2\pi)^3 i}{2}\int d^4 l\prod_{i=0}^{2}\delta(l_i^2)\langle K_2^-|\not{l}|a^-\rangle\\
&\qquad = -\frac{(2\pi)^3 i}{2}\int dt J_t'\frac{[\chi K_1^\flat][K_2 a]}{[K_1^\flat K_2]^2}\left(\langle K_2^-|\not{K}_1|K_2^-\rangle t + \frac{S_1}{\gamma}\langle\chi^-|\not{K}_2|K_1^{\flat,-}\rangle\right),
\end{aligned}
\tag{5.21}
$$

where we have used the parameterisation of l^μ given by eq. (B9) and added an extra overall factor of i which would come from the additional tree amplitude in a triple cut.

Of this triple cut integrand only the first, t dependent, term can give anything other than a scalar triangle contribution. To derive the result of this integral over t we will, as we have done previously, use our parameterisation of the cut momentum, eq. (5.9), to pick out the integral as follows

$$
-i\int d^4 l\frac{\langle\chi^-|\not{l}|K_1^{\flat,-}\rangle}{l^2 l_1^2 (l+K_2)^2} \equiv (2\pi)^3\gamma\int dt J_t' t. \tag{5.22}
$$

Using Passarino-Veltman reduction on the single tensor integral on the left hand side of this as well as dropping anything but the contributing bubble integrals of our particular cut leaves us with the result

$$
\int dt J_t' t = \frac{2}{(2\pi)^3}\frac{S_1\langle\chi K_2\rangle[K_2 K_1^\flat]}{\gamma\langle K_2^-|\not{K}_1|K_2^-\rangle^2}B_0^{\text{cut}}(K_1^2), \tag{5.23}
$$

where $B_0^{\text{cut}}(K_1^2)$ is the cut form of the scalar bubble integral of eq. (C1). This non-vanishing result for the integral over t, in contrast to that of section IV A, is a direct consequence of the cut momentum parameterisation forced upon us when taking the residues contained in the two-particle cut integrand with which we started.

On substituting the result of eq. (5.23) into eq. (5.21) we find that we can write eq. (5.20), using the bubble integral given in eq. (C1), as

$$
\begin{aligned}
(2\pi)^2\int d^4 l\prod_{i=0}^{1}\delta(l_i^2)\frac{\langle K_2^-|\not{l}|a^-\rangle}{(l+K_2)^2} &= -i\frac{[\chi K_1^\flat][K_2 a]}{[K_1^\flat K_2]^2}\frac{S_1}{\gamma}\frac{\langle\chi^-|\not{K}_2|K_1^{\flat,-}\rangle}{\langle K_2^-|\not{K}_1|K_2^-\rangle}B_0^{\text{cut}}(K_1^2) + i\frac{[K_1^\flat a]}{[K_1^\flat K_2]}B_0^{\text{cut}}(K_1^2)\\
&= i\frac{\langle K_2^-|\not{K}_1|a^-\rangle}{\langle K_2^-|\not{K}_1|K_2^-\rangle}B_0^{\text{cut}}(K_1^2),
\end{aligned}
\tag{5.24}
$$

which is the known coefficient of the scalar bubble contained inside the linear triangle.

Of course, if we had chosen $\chi = K_2$ from the beginning, then the first term on the left hand side of eq. (5.20) would have been the complete bubble coefficient. In general, if we are able to rewrite a two-particle cut integrand such that each term contains only a single propagator then we can always choose a different $\chi = K_2^\flat$, defined via

$$K_2^{\flat,\mu} = K_2^\mu - \frac{S_2}{\langle K_1^{\flat,-}|\not{K}_2|K_1^{\flat,-}\rangle} K_1^{\flat,\mu}, \tag{5.25}$$

for each term individually such that there are no contributions from the residue terms. Whether this is both feasible and a more computationally effective approach than calculating the residue contributions through the use of triple cuts would depend upon the cut integrand in question.

In general we will be considering processes which contain terms with powers of up to t^3, so we will need to know these integrals. Again these can be found using a straightforward application of tensor reduction techniques. When all three legs in the cut are massive these integrals over t are given, after dropping an overall factor of $1/(2\pi)^3$ witch always cancels out of the final coefficient, by

$$T(j) = \int dt J_t' t^j = \left(\frac{S_1}{\gamma}\right)^j \frac{\langle \chi^-|\not{K}_2|K_1^{\flat,-}\rangle^j (K_1\cdot K_2)^{j-1}}{\Delta^j} \left(\sum_{l=1}^{j} \mathcal{C}_{jl} \frac{S_2^{l-1}}{(K_1\cdot K_2)^{l-1}}\right) B_0^{\text{cut}}(K_1^2). \tag{5.26}$$

Simply taking the relevant mass to zero gives the forms in the one and two mass cases. Δ was previously defined in eq. (4.15) and we have

$$\begin{aligned}
\mathcal{C}_{11} &= \frac{1}{2}, \\
\mathcal{C}_{21} &= -\frac{3}{8}, \quad \mathcal{C}_{22} = -\frac{3}{8}, \\
\mathcal{C}_{31} &= -\frac{1}{12}\frac{\Delta}{(K_1\cdot K_2)^2} + \frac{5}{16}, \quad \mathcal{C}_{32} = \frac{5}{8}, \quad \mathcal{C}_{33} = \frac{5}{16}.
\end{aligned} \tag{5.27}$$

Also for later use we define $T(0) = 0$.

E. The bubble coefficient

We have now assembled all the pieces necessary to compute our desired scalar bubble coefficient, b_j, corresponding to the cut scalar bubble integral $B_0^{\text{cut}}(K_j^2)$. It is given in general not as the coefficient of a single term but by summing together the $t^0 y^m$ terms from both the

double Inf in y followed by t as well as residue contributions which we derive by considering all possible triple cuts contained in the two-particle cut. The coefficient is given by

$$b_j = -i \left[\mathrm{Inf}_t \left[\mathrm{Inf}_y A_1 A_2\right](y)\right](t)\Big|_{t\to 0,\, y^m \to \frac{1}{m+1}} - \frac{1}{2}\sum_{\{\mathcal{C}_{\mathrm{tri}}\}} \left[\mathrm{Inf}_t A_1 A_2 A_3\right](t)\Big|_{t^j \to T(j)}, \qquad (5.28)$$

where $T(j)$ is defined in eq. (5.26) and the sum over the set $\{\mathcal{C}_{\mathrm{tri}}\}$ is a sum over all triple cuts obtainable by cutting one more leg of the two-particle cut integrand A_1A_2.

When computing with eq. (5.28) there is a freedom in the choice of χ. A suitable choice of which can simplify the degree of computation involved in extracting a particular coefficient. Particular choices of χ can eliminate the need to calculate the second term of eq. (5.28) completely, as discussed in section V D. We also note that there are choices of χ which eliminate the need to evaluate the first term of eq. (5.28), so that the coefficient comes entirely from the second term of eq. (5.28) instead.

VI. APPLICATIONS

To demonstrate our method we now present the recalculation of some representative triangle and bubble integral coefficients. We also discuss checks we have made against other various state-of-the-art cut-constructable coefficients contained in the literature.

A. Extracting coefficients

To highlight the application of our procedure to the extraction of basis integral coefficients we consider deriving the coefficients of some simple integral functions which commonly appear, for example, in one-loop Feynman diagrams.

1. *The triangle coefficient of a linear two-mass triangle*

First we consider deriving the scalar triangle coefficient of a linear two-mass triangle with massive leg K_1, massless leg K_2, and a and b arbitrary massless four-vectors not equal to K_2. This is given by the integral

$$-i\int d^4l \frac{\langle a^-|\not{l}|b^-\rangle}{l^2(l-K_1)^2(l+K_2)^2}. \qquad (6.1)$$

Extracting the triangle coefficient requires cutting all three propagators of the integrand. We do this here by simply removing the "cut" propagator as we are interested only in the integrand. This leaves only

$$\langle a^-|\not{l}|b^-\rangle. \tag{6.2}$$

Rewriting this integrand in terms of the parameterisation of eq. (4.16) gives

$$\left(\alpha_{01}\langle a^-|\not{K}_2|b^-\rangle + t\langle aK_1^\flat\rangle[\chi b]\right). \tag{6.3}$$

As $S_2 = 0$ we see that $\alpha_{01} = S_1/\gamma$ and that $\gamma = 2(K_1 \cdot K_2)$. Then taking the t^0 component of the $[\mathrm{Inf}_t]$ of this in accordance with eq. (4.10) leaves us with our desired coefficient

$$-\frac{S_1}{\langle K_2^-|\not{K}_1|K_2^-\rangle}\langle a^-|\not{K}_2|b^-\rangle, \tag{6.4}$$

which matches the expected result.

2. *The bubble contributions of a three-mass linear triangle*

Consider a linear triangle with in this case three massive legs, so now K_2 is massive but again a and b are arbitrary massless four-vectors,

$$-i\int d^4l\frac{\langle a^-|\not{l}|b^-\rangle}{l^2(l-K_1)^2(l+K_2)^2}. \tag{6.5}$$

Extracting the bubble coefficient of the integral $B_0(K_1^2)$ is done by cutting the two propagators l^2 and $(l-K_1)^2$. Again cutting the legs is done by removing the relevant propagators from the integrand so that it is given by

$$\frac{\langle a^-|\not{l}|b^-\rangle}{(l+K_2)^2}. \tag{6.6}$$

As this contains a single propagator, and therefore a single pole, we could choose to set $\chi = K_2^\flat$ (as defined in eq. (5.25)), before performing the series expansions in y and t. For this choice of χ the bubble coefficient comes entirely from the two-particle cut. Using the first term of eq. (5.28) gives directly

$$-i\left(\frac{\gamma\langle a^-|\not{K}_1^\flat|b^-\rangle}{\gamma^2 - S_1S_2} - \frac{S_1\langle a^-|\not{K}_2^\flat|b^-\rangle}{\gamma^2 - S_1S_2}\right), \tag{6.7}$$

where $\gamma = \langle K_2^{\flat,-}|\not{K}_1^\flat|K_2^{\flat,-}\rangle$, a result which is equivalent to the expected answer.

In order to demonstrate the procedure of using triple cut contributions in extracting a bubble coefficient we will now reproduce this by assuming $\chi \neq K_2^\flat$. For this case the first term of eq. (5.28) then gives

$$- i \frac{\langle a\chi\rangle [K_1^\flat b]}{\langle \chi^- | \not{K}_2 | K_1^{\flat,-}\rangle}, \tag{6.8}$$

which upon choosing $\chi = a$ vanishes and so the complete contribution will come from the triple cut pieces of eq. (6.5). Cutting the remaining propagator in eq. (6.6) gives us the single triple cut term which will contribute. The integrand of this is given, after multiplying by an additional factor of i which would come from the third tree amplitude if this was a triple cut, by

$$i \left(\langle al\rangle [lb] \Big|_{y=y_+} + \langle al\rangle [lb] \Big|_{y=y_-} \right) = i \left((y_+ + y_-) \langle a K_1^\flat\rangle [K_1^\flat b] + 2t \langle a K_1^\flat\rangle [ab] \right), \tag{6.9}$$

where we have set $\chi = a$. From eq. (5.16) we have

$$y_+ + y_- = \frac{\gamma \left(\langle K_1^{\flat,-} | \not{K}_2 | K_1^{\flat,-}\rangle - S_1 \langle a^- | \not{K}_2 | a^-\rangle \right) t + S_1 \langle a^- | \not{K}_2 | K_1^{\flat,-}\rangle}{S_1 \langle a^- | \not{K}_2 | K_1^{\flat,-}\rangle}, \tag{6.10}$$

Hence taking the $[\mathrm{Inf}_t]$ of the cut integrand, eq. (6.9), and dropping any terms not proportional to t leaves

$$it \langle a^- | \not{K}_1 | b^-\rangle \left(\frac{\gamma \langle K_1^{\flat,-} | \not{K}_2 | K_1^{\flat,-}\rangle - S_1 \langle a^- | \not{K}_2 | a^-\rangle}{S_1 \langle a^- | \not{K}_2 | K_1^{\flat,-}\rangle} + 2 \frac{[ab]}{[K_1^\flat b]} \right), \tag{6.11}$$

which after inserting the result for the t integral given by eq. (5.26) and substituting this into the second term of eq. (5.28) gives for our desired coefficient

$$\begin{aligned} & -i \frac{\langle a^- | \not{K}_1 | b^-\rangle}{4\,\Delta} \left(\langle K_1^{\flat,-} | \not{K}_2 | K_1^{\flat,-}\rangle - \frac{S_1}{\gamma} \langle a^- | \not{K}_2 | a^-\rangle + \frac{2 S_1}{\gamma} \langle a^- | \not{K}_2 | K_1^{\flat,-}\rangle \frac{[ab]}{[K_1^\flat b]} \right) \\ & = -i \frac{1}{2\Delta} \left((K_1 \cdot K_2) \langle a^- | \not{K}_1 | b^-\rangle - S_1 \langle a^- | \not{K}_2 | b^-\rangle \right), \end{aligned} \tag{6.12}$$

where Δ was given in eq. (4.15). This matches both the expected result and eq. (6.7).

B. Constructing the one-loop six-photon amplitude $A_6(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$

Recently an analytic form for the last unknown six-photon one-loop amplitude was obtained by Binoth, Heinrich, Gehrmann and Mastrolia in ref. [46]. This result was used

to confirm a previous numerical result [50]. More recently still further corroboration has been provided by [45]. Here we reproduce, as an example, the calculation of the three-mass triangle and bubble coefficients, again confirming part of these results.

Firstly it is a very simple exercise to demonstrate by explicit computation that all bubble coefficients vanish. If we were to use the basis of finite box integrals, as defined in [35], then there is only a single unique three-mass triangle coefficient, a complete explicit derivation of which we now present. Starting from the cut in the $12:34:56$ channel shown in figure 4 we can write the cut integrand as

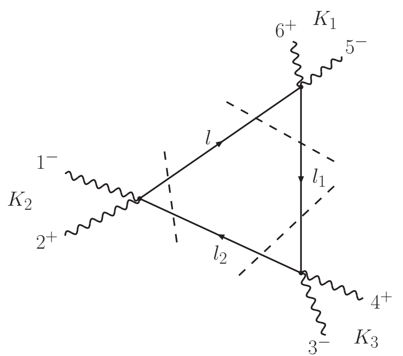

FIG. 4: Triple cut six-photon amplitude in the $12:34:56$ channel.

$$16A_4(-l_q^{-h},1^-,2^+,l_{2,\bar{q}}^{h_2})A_4(-l_{2,q}^{-h_2},3^-,4^+,l_{1,\bar{q}}^{h_1})A_4(-l_{1,q}^{-h_1},5^-,6^+,l_{\bar{q}}^{h}), \tag{6.13}$$

with all unlabelled legs photons and $l_1 = l - K_{56}$ and $l_2 = l + K_{12}$. The overall factor of 16 comes from the differing normalisation conventions between QCD colour-ordered amplitudes and QED photon amplitudes. Both helicity choices $h = h_1 = h_2 = \pm$ give identical contributions. Written explicitly, eq. (6.13) is

$$32i\frac{\langle l1\rangle^2\langle l_2 3\rangle^2\langle l_1 5\rangle^2}{\langle l2\rangle\langle l_2 2\rangle\langle l_1 4\rangle\langle l_2 4\rangle\langle l6\rangle\langle l_1 6\rangle}. \tag{6.14}$$

After inserting the momentum parameterisation of eq. (4.16) this becomes

$$\begin{aligned}&32i\frac{1}{\left(t\langle K_1^\flat 2\rangle+\alpha_{01}\langle K_2^\flat 2\rangle\right)\left(t\langle K_1^\flat 2\rangle+\alpha_{21}\langle K_2^\flat 2\rangle\right)\left(t\langle K_1^\flat 4\rangle+\alpha_{11}\langle K_2^\flat 4\rangle\right)}\\ &\qquad\times\frac{\left(t\langle K_1^\flat 1\rangle+\alpha_{01}\langle K_2^\flat 1\rangle\right)^2\left(t\langle K_1^\flat 3\rangle+\alpha_{21}\langle K_2^\flat 3\rangle\right)^2\left(t\langle K_1^\flat 5\rangle+\alpha_{11}\langle K_2^\flat 5\rangle\right)^2}{\left(t\langle K_1^\flat 4\rangle+\alpha_{21}\langle K_2^\flat 4\rangle\right)\left(t\langle K_1^\flat 6\rangle+\alpha_{01}\langle K_2^\flat 6\rangle\right)\left(t\langle K_1^\flat 6\rangle+\alpha_{11}\langle K_2^\flat 6\rangle\right)}.\end{aligned} \tag{6.15}$$

Applying eq. (4.10) implies taking only the t^0 piece of the $[\mathrm{Inf}_t]$ of this expression. Averaging over both solutions leaves us with our form for the three mass triangle coefficient

$$-16i\sum_{\gamma=\gamma_\pm}\frac{\langle K_1^\flat 1\rangle^2\langle K_1^\flat 3\rangle^2\langle K_1^\flat 5\rangle^2}{\langle K_1^\flat 2\rangle^2\langle K_1^\flat 4\rangle^2\langle K_1^\flat 6\rangle^2}, \tag{6.16}$$

where $K_1^\flat$ depends upon the form of $\gamma_\pm$ as given in eq. (4.14). Numerical comparison with the analytic result of [46] shows complete agreement.

C. Contributions to the one-loop $A_6(1_q^+, 2_{\overline{q}}^-, 3^-, 4^+; 5_{\overline{e}}^-, 6_e^+)$ amplitude

This particular amplitude was originally obtained by Bern, Dixon and Kosower in [19]. Making up this amplitude are many box, triangle and bubble integrals along with rational terms. Here we will recompute one particular representative three-mass triangle coefficient in order to highlight the application of our technique to a phenomenologically interesting process.

Following the notation of [19], we wish to calculate the three-mass triangle coefficient of $I_3^{3m}(s_{14}, s_{23}, s_{56}) \equiv C_0(s_{14}, s_{56})$ of the F^{cc} term. The only contributing cut is shown in figure 5. We begin by writing down the triple cut integrand for this case

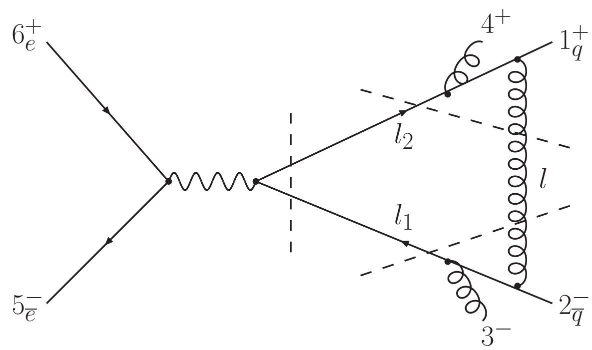

FIG. 5: Triple cut in the 14 : 23 : 56 channel.

$$A_4(-l_{1,\bar{q}}^{-h_1}, 5_{\overline{e}}^-, 6_e^+, l_{2,q}^{h_2})A_4(-l_{2,\bar{q}}^{-h_2}, 4^+, 1_q^+, l_g^h)A_4(-l_g^{-h}, 2_{\overline{q}}^-, 3^-, l_{1,q}^{h_1}), \tag{6.17}$$

where $l_1 = l - K_{23}$ and $l_2 = l + K_{14}$. Only when $h = -$, $h_1 = +$ and $h_2 = +$ do we get a contribution. It can be written explicitly as

$$i\frac{\langle l_2 5\rangle^2 \langle l l_2\rangle^2 \langle 23\rangle^2}{\langle 14\rangle\langle 56\rangle\langle 4 l_2\rangle\langle 2l\rangle\langle l l_1\rangle\langle l_1 l_2\rangle}. \tag{6.18}$$

Rewriting this in terms of the loop momentum parametrisation of eq. (4.16) gives

$$i\frac{\gamma\left(t\langle K_1^\flat 5\rangle + \alpha_{21}\langle K_2^\flat 5\rangle\right)^2 \langle 23\rangle^2}{s_{23}\left(1 - \frac{s_{23}}{\gamma}\right)\langle 14\rangle\langle 56\rangle\left(t\langle 4K_1^\flat\rangle + \alpha_{21}\langle 4K_2^\flat\rangle\right)\left(t\langle 2K_1^\flat\rangle + \alpha_{01}\langle 2K_2^\flat\rangle\right)}. \tag{6.19}$$

The two solutions of γ are given by $\gamma_\pm = -(K_{23}\cdot K_{14}) \pm \sqrt{(K_{23}\cdot K_{14})^2 - s_{23}s_{14}}$, the α_{ij}'s are given in Appendix A.

The application of eq. (4.10) involves taking [Inf_t] of eq. (6.19), dropping all but the t^0 component of the result and then averaging over both solutions of γ giving the coefficient

$$-\frac{i}{2}\sum_{\gamma=\gamma_\pm}\frac{\gamma\langle K_1^\flat 5\rangle^2\langle 23\rangle^2}{S_1\left(1-\frac{S_1}{\gamma}\right)\langle 14\rangle\langle 56\rangle\langle 4K_1^\flat\rangle\langle 2K_1^\flat\rangle}, \tag{6.20}$$

where again $K_1^\flat$ depends upon $\gamma_\pm$. Numerical comparison against the solution for this coefficient presented in [19],

$$-\frac{i}{2}\frac{[14]\left(\langle 2^-|\not{K}_{14}\not{K}_{23}|5^+\rangle^2 - \langle 25\rangle^2 s_{14}s_{23}\right)}{\langle 14\rangle[23]\langle 56\rangle\langle 2^-|\not{K}_{14}|3^-\rangle\langle 2^-|\not{K}_{34}|1^-\rangle} + \text{flip}, \tag{6.21}$$

shows complete agreement, where the operation flip is defined as the exchanges $1 \leftrightarrow 2$, $3 \leftrightarrow 4$, $5 \leftrightarrow 6$, $\langle ab\rangle \leftrightarrow [ab]$.

The remaining triangle and bubble coefficients can be derived in an analogous way. We have computed a selection of these coefficients for $A_6(1_q^+, 2_{\bar{q}}^-, 3^-, 4^+; 5_{\bar{e}}^-, 6_e^+)$, along with coefficients of other amplitudes given in [19], and find complete agreement.

D. Bubble coefficients of the one-loop 5-gluon QCD amplitude $A_5(1^-, 2^-, 3^+, 4^+, 5^+)$

This result for the 1-loop 5 gluon QCD amplitude $A_5(1^-, 2^-, 3^+, 4^+, 5^+)$ was originally calculated by Bern, Dixon, Dunbar and Kosower in [18]. It contains neither box nor triangle integrals, only bubbles. We need therefore only compute bubble coefficients. There are only a pair of such coefficients, with masses s_{23} and $s_{234} = s_{51}$.

For the first cut in the channel $K_1 = K_{23}$ we have, for the sum of the two possible helicity configurations, the two-particle cut integrand

$$\frac{2}{\langle 23\rangle\langle 45\rangle\langle 51\rangle}\frac{\langle 1l_1\rangle^2\langle 1l\rangle\langle 2l\rangle\langle 2l_1\rangle^2}{\langle 4l_1\rangle\langle 3l_1\rangle\langle ll_1\rangle^2}, \tag{6.22}$$

and for the second, in the channel $K_1 = K_{234}$,

$$\frac{2}{\langle 23\rangle\langle 34\rangle\langle 51\rangle}\frac{\langle 1l_1\rangle^2\langle 1l\rangle\langle 2l\rangle\langle 2l_1\rangle^2}{\langle 4l_1\rangle\langle 5l_1\rangle\langle ll_1\rangle^2}. \tag{6.23}$$

Focus upon the $K_1 = K_{23}$ cut initially. There are two pole-containing terms in the denominator of this cut. We could choose to partial fraction these terms and then pick $\chi =$

K_2 in each case to extract the coefficient. Instead though we will derive the coefficient using triple cut contributions. Choosing $\chi = k_1$ so that after inserting the cut loop momentum parameterisation of eq. (5.8) the cut integrand becomes

$$\frac{2\gamma^2\langle 1K_1^\flat\rangle}{S_1^2\langle 23\rangle\langle 45\rangle\langle 51\rangle} t \frac{\left(\langle 2K_1^\flat\rangle - \frac{S_1}{\gamma}\frac{y}{t}\langle 21\rangle\right)\left(t\langle 2K_1^\flat\rangle + \frac{S_1}{\gamma}\left(1-y\right)\langle 21\rangle\right)}{\left(\langle 3K_1^\flat\rangle - \frac{S_1}{\gamma}\frac{y}{t}\langle 31\rangle\right)\left(\langle 4K_1^\flat\rangle - \frac{S_1}{\gamma}\frac{y}{t}\langle 41\rangle\right)}, \tag{6.24}$$

and hence produces no $[\mathrm{Inf}_y[\mathrm{Inf}_t]]$ term. Consequentially the two-particle cut contribution to the bubble coefficient vanishes. The same choice of χ similarly removes all two-particle cut contributions in the channel $K_1 = K_{234}$ from the corresponding scalar bubble coefficient.

Examining the triple cuts of the bubble in the K_{23} channel shows only two possible contributions, again after summing over both contributing helicities, given by

$$\frac{2i}{\langle 45\rangle\langle 51\rangle}\frac{[3l][3l_2]\langle 1l_1\rangle\langle 1l\rangle^2\langle 2l_1\rangle\langle 2l\rangle}{\langle ll_1\rangle\langle l_1l_2\rangle[ll_2]\langle l4\rangle}, \tag{6.25}$$

when $K_2 = k_3$ and

$$-\frac{2i}{\langle 23\rangle\langle 51\rangle}\frac{[4l][4l_2]\langle 1l_1\rangle\langle 1l_2\rangle^2\langle 2l_1\rangle\langle 2l\rangle^2}{\langle ll_1\rangle\langle l_1l_2\rangle[ll_2]\langle 5l_2\rangle\langle 3l\rangle}, \tag{6.26}$$

when $K_2 = k_4$. In both cases K_2 is massless and is of positive helicity so we use the parameterisation of the triple cut momenta for y_+ given in eq. (B2). Then along with setting $\chi = k_1$ gives for the first triple cut integrand

$$\frac{2i\langle 1K_1^\flat\rangle^2\langle 23\rangle}{\langle 13\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}\frac{\gamma^2}{S_1^2}t\left(\frac{\langle 1^-|\not{2}|3^-\rangle}{\langle 1K_1^\flat\rangle} - \frac{\gamma t}{S_1}\frac{\langle 3^-|\not{K}_{23}|3^-\rangle}{\langle 13\rangle}\right)\left(t\frac{\langle 1K_1^\flat\rangle}{\langle 13\rangle}\langle 23\rangle + \frac{S_1}{\gamma}\langle 21\rangle\right), \tag{6.27}$$

and for the second

$$-\frac{2i\langle 1K_1^\flat\rangle^2\langle 24\rangle^2}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle\langle 14\rangle}\frac{\gamma^2}{S_1^2}t\left(\frac{\gamma t}{S_1}\frac{\langle 4^-|\not{K}_{23}|4^-\rangle}{\langle 14\rangle} - \frac{\langle 1^-|\not{K}_{23}|4^-\rangle}{\langle 1K_1^\flat\rangle}\right)\left(t\frac{\langle 1K_1^\flat\rangle}{\langle 14\rangle}\langle 24\rangle + \frac{S_1\langle 21\rangle}{\gamma}\right). \tag{6.28}$$

Applying these integrands to the second term of eq. (5.28) by taking $[\mathrm{Inf}_t]$, dropping any terms not proportional to t and then performing the substitution $t^i \to T(i)$ gives for the coefficient of the first triple cut simply $\frac{1}{3}A_5^{\mathrm{tree}}$, and for the second triple cut

$$-\frac{A_5^{\mathrm{tree}}}{s_{12}^3}\frac{\langle 1^+|\not{2}\not{4}\not{K}_{23}|1^+\rangle^2}{\langle 4^-|\not{K}_{23}|4^-\rangle^2}\left(s_{12} - \frac{2}{3}\frac{\langle 1^+|\not{2}\not{4}\not{K}_{23}|1^+\rangle}{\langle 4^-|\not{K}_{23}|4^-\rangle}\right). \tag{6.29}$$

After following the same series of steps as above for the second bubble coefficient with $K_1 = K_{234}$ we find only a single triple cut contributing term corresponding to $K_2 = k_4$. This

is related to the second triple cut coefficient derived above via the replacement $K_{23} \to K_{234}$ and swapping the overall sign.

After combining the three triple cut pieces above we arrive at the following form for the cut constructable pieces of this amplitude

$$\frac{r_\Gamma(\mu^2)^\epsilon}{(4\pi)^{2-\epsilon}}\Bigg(\frac{1}{3}A_5^{\text{tree}}B_0(s_{23}) \\ - \frac{A_5^{\text{tree}}}{s_{12}^3}\frac{\langle 1^+|\not{2}\not{4}\not{K}_{23}|1^+\rangle^2}{\langle 4^-|\not{K}_{23}|4^-\rangle^2}\left(s_{12} - \frac{2}{3}\frac{\langle 1^+|\not{2}\not{4}\not{K}_{23}|1^+\rangle}{\langle 4^-|\not{K}_{23}|4^-\rangle}\right)(B_0(s_{23}) - B_0(s_{234}))\Bigg), \quad (6.30)$$

which can easily be shown to match the result given in [18].

While this example is particularly simple we have also performed additional comparisons against other results in the literature. Such tests include the cut constructible pieces of all two-minus gluon amplitudes with up to seven external legs, originally obtained in [18, 37]. Additionally we find agreement for the case when, with six gluon legs, three are of negative helicity and adjacent to each other and the remainder are positive helicity, which was originally obtained in [49]. We have also successfully reproduced the known three mass triangle coefficients in $\mathcal{N} = 1$ supersymmetry for $A_6(1^-, 2^+, 3^-, 4^+, 5^-, 6^+)$ and $A_6(1^-, 2^-, 3^+, 4^-, 5^+, 6^+)$, originally obtained in [35].

VII. CONCLUSIONS

The calculation of Standard Model background processes at the LHC requires efficient techniques for the production of amplitudes. The large numbers of processes involved along with their differing partonic makeups suggests that as much automation as possible is desired. In this paper we have presented a new formalism which directs us towards this goal. Coefficients of the basis scalar integrals making up a one-loop amplitude are constructed in a straightforward manner involving only a simple change of variables and a series expansion, thus avoiding the need to perform any integration or calculate any extraneous intermediate quantities. The main results of this paper can be encapsulated simply by eq. (4.10) and eq. (5.28) along with the cut loop momentum given by eq. (4.16), eq. (5.8) and eq. (5.16).

Although this technique has been presented mainly in the context of using generalised unitarity [19, 39, 40, 41] to construct coefficients, and hence the cut-constructible part of the amplitude, it can also be used as an efficient method of performing one-loop integration. Using the idea of "cutting" two, three or four of the propagators inside an integral, we

isolate and then extract scalar basis coefficients. This procedure then allows us to rewrite the integral in terms of the scalar one-loop basis integrals, hence giving us a result for the integral.

Different unitarity cuts isolate particular basis integrals. For the extraction of triangle integral coefficients this means triple cuts and for bubble coefficients we use a combination of two-particle and triple cuts. Extracting the desired coefficients from these cut integrands is then a two step process. The first step is to rewrite the cut loop momentum in terms of a parameterisation which depends upon the remaining free parameters of the integral after all the cut delta functions have been applied. Triangle coefficients are then found by taking the terms independent of the sole free integral parameter as this parameter is taken to infinity.

Bubble coefficients are calculated in a similar if slightly more complicated way. The presence of a second free parameter in the bubble case means that we must take into account, not only the constant term in the expansion of the cut integrand as the free integral parameters are taken to infinity, but also powers of one of these parameters. The limit on the maximum power of l^μ appearing in the cut integral restricts the appearance of such terms and hence we need consider only finite numbers of powers of these free parameters. Additionally it can also be necessary to take into account contributions from terms generated by applying an additional cut to the bubble integral. The flexibility in our choice of the cut-loop momentum parameterisation allows us to directly control whether we need compute any of these triple cut terms. Furthermore we can control which of these triple cut terms appears, in cases when their computation is necessary.

As we consider the application of this procedure to more diverse processes than those detailed here, we should also investigate the "complexity" of the generated coefficients. In the applications we have presented we can see that we produce "compact" forms with minimal amounts of simplification required. This is important if we are to consider further automation. The straightforward nature of this technique combined with the minimal need for simplification means that efficient computer implementations can easily be produced. As a test of this assertion we have implemented the formalism within a `Mathematica` program which has been used to perform checks against state-of-the-art results contained in the literature. Such checks have included various helicity configurations of up to seven external gluons as well as the bubble and three-mass triangle coefficients of the six photon $A_6(-+-+-+)$ amplitude. In addition representative coefficients of processes of the type $e^+e^- \to q\bar{q}gg$

have been successfully obtained.

Our procedure as presented has mainly been in the context of massless theories. Fundamentally there is no restriction to the application of this to theories also involving massive fields circulating in the loop. Extensions to include masses should require only a suitable momentum parameterisation for the cut loop momentum; the procedure is then expected to apply as before.

In conclusion therefore we believe that the technique presented here shows great potential for easing the calculation of needed one-loop integrals for current and future colliders.

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APPENDIX A: THE TRIPLE CUT PARAMETERISATION

In this appendix we give the complete detail of the triple cut parameterisation along with some other useful results. The three cut momenta are given by

$$\langle l_i^-| = t\langle K_1^{\flat,-}| + \alpha_{i1}\langle K_2^{\flat,-}|, \quad \langle l_i^+| = \frac{\alpha_{i2}}{t}\langle K_1^{\flat,+}| + \langle K_2^{\flat,+}|, \tag{A1}$$

with

$$\begin{aligned}
\alpha_{01} &= \frac{S_1(\gamma - S_2)}{(\gamma^2 - S_1S_2)}, & \alpha_{02} &= \frac{S_2(\gamma - S_1)}{(\gamma^2 - S_1S_2)}, \\
\alpha_{11} = \alpha_{01} - \frac{S_1}{\gamma} &= -\frac{S_1S_2(1 - (S_1/\gamma))}{\gamma^2 - S_1S_2}, & \alpha_{12} = \alpha_{02} - 1 &= \frac{\gamma(S_2 - \gamma)}{\gamma^2 - S_1S_2}, \\
\alpha_{21} = \alpha_{01} - 1 &= \frac{\gamma(S_1 - \gamma)}{\gamma^2 - S_1S_2}, & \alpha_{22} = \alpha_{02} - \frac{S_2}{\gamma} &= -\frac{S_1S_2(1 - (S_2/\gamma))}{\gamma^2 - S_1S_2},
\end{aligned} \tag{A2}$$

along with the identities $\alpha_{01}\alpha_{02} = \alpha_{11}\alpha_{12}$ and $\alpha_{01}\alpha_{02} = \alpha_{21}\alpha_{22}$. When written as four-vectors the cut momentum are given by

$$l_i^\mu = \alpha_{i2}K_1^{\flat,\mu} + \alpha_{i1}K_2^{\flat,\mu} + \frac{t}{2}\langle K_1^{\flat,-}|\gamma^\mu|K_2^{\flat,-}\rangle + \frac{\alpha_{i1}\alpha_{i2}}{2t}\langle K_2^{\flat,-}|\gamma^\mu|K_1^{\flat,-}\rangle. \tag{A3}$$

From these parameterised forms we have the following spinor product identities

$$\begin{aligned}
[l l_1] &= \frac{\alpha_{12}-\alpha_{02}}{t}[K_1^\flat K_2^\flat] = -\frac{1}{t}[K_2^\flat K_1^\flat],\\
\langle l l_1\rangle &= t(\alpha_{11}-\alpha_{01})\langle K_1^\flat K_2^\flat\rangle = -\frac{tS_1}{\gamma}\langle K_1^\flat K_2^\flat\rangle,\\
[l l_2] &= \frac{\alpha_{22}-\alpha_{02}}{t}[K_1^\flat K_2^\flat] = -\frac{S_2}{\gamma t}[K_2^\flat K_1^\flat],\\
\langle l l_2\rangle &= t(\alpha_{21}-\alpha_{01})\langle K_1^\flat K_2^\flat\rangle = -t\langle K_1^\flat K_2^\flat\rangle,\\
[l_1 l_2] &= \frac{\alpha_{22}-\alpha_{12}}{t}[K_1^\flat K_2^\flat] = \frac{1}{t}\left(1-\frac{S_2}{\gamma}\right)[K_2^\flat K_1^\flat],\\
\langle l_1 l_2\rangle &= t(\alpha_{11}-\alpha_{21})\langle K_1^\flat K_2^\flat\rangle = -t\left(1-\frac{S_1}{\gamma}\right)\langle K_1^\flat K_2^\flat\rangle.
\end{aligned} \tag{A4}$$

and we note that

$$-\left(1-\frac{S_2}{\gamma}\right)\left(1-\frac{S_1}{\gamma}\right)\gamma = -\gamma - \frac{S_1S_2}{\gamma} + S_1 + S_2 = (K_1-K_2)^2 = S_3, \tag{A5}$$

and so with $l \equiv l_0$ we have $\langle l_i l_j\rangle[l_j l_i] = S_{i+j}$, as expected.

APPENDIX B: THE TRIPLE CUT BUBBLE CONTRIBUTION MOMENTUM PARAMETERISATION WHEN $K_2^2 = 0$

In this appendix we give the forms for the triple cut momentum of section V C in the case when $S_2 = 0$, i.e. we have a one or two mass triangle. Firstly in these cases the K_2 leg is attached to a three-point vertex and so the amplitude for this will contain either $[K_2 l]$ or $\langle K_2 l\rangle$ depending upon the helicity of K_2. This means that in the positive helicity case only the delta function solution $\delta(y-y_+)$ survives and for a negative helicity K_2 the $\delta(y-y_-)$ survives. We have for both solutions

$$J'_t = \frac{1}{\left(S_1\langle\chi^-|\not{K}_2|K_1^{\flat,-}\rangle + t\gamma\langle K_2^-|\not{K}_1|K_2^-\rangle\right)}. \tag{B1}$$

The momentum parameterisation for the y_+ solution is given in spinor components by

$$\langle l^-| = \frac{\langle\chi K_1^\flat\rangle}{\langle\chi K_2\rangle}\langle K_2^-|, \qquad \langle l^+| = \langle K_1^{\flat,+}| - \frac{\gamma t}{S_1}\frac{1}{\langle\chi K_2\rangle}\langle K_2^-|\not{K}_1, \tag{B2}$$

and as a 4-vector by

$$l^\mu = \frac{\langle\chi K_1^\flat\rangle}{2\langle\chi K_2\rangle}\left(\frac{t\gamma}{S_1\langle\chi K_2\rangle}\langle K_2^-|\gamma^\mu\not{K}_1|K_2^+\rangle + \langle K_2^-|\gamma^\mu|K_1^{\flat,-}\rangle\right). \tag{B3}$$

The other momenta are given by

$$\langle l_1^-| = t\frac{\langle\chi K_1^\flat\rangle}{\langle\chi K_2\rangle}\langle K_2^-| - \frac{S_1}{\gamma}\langle\chi^-|, \qquad \langle l_1^+| = -\frac{\gamma}{S_1\langle\chi K_2\rangle}\langle K_2^-|\not{K}_1,$$
$$\langle l_2^-| = \frac{\langle\chi K_1^\flat\rangle}{\langle\chi K_2\rangle}\langle K_2^-|, \qquad \langle l_2^+| = -\frac{\langle\chi^-|\not{K}_3}{\langle\chi K_1^\flat\rangle} - \frac{\gamma t}{S_1}\frac{1}{\langle\chi K_2\rangle}\langle K_2^-|\not{K}_1. \tag{B4}$$

where we have moved the overall factor of t from $\langle l_1^-|$ to $\langle l_1^+|$ to avoid the presence of a $1/t$ term for aesthetical reasons. The spinor products formed from these are given by

$$\langle ll_1\rangle = \frac{S_1}{\gamma}\langle\chi K_1^\flat\rangle, \qquad [ll_1] = [K_1^\flat\chi],$$
$$\langle ll_2\rangle = 0, \qquad [ll_2] = -\frac{\langle\chi K_2\rangle}{\langle\chi K_1^\flat\rangle}[lK_2],$$
$$\langle l_1 l_2\rangle = -\frac{S_1}{\gamma}\langle\chi K_1^\flat\rangle, \qquad [l_1 l_2] = -\frac{S_3}{S_1}[K_1^\flat\chi]. \tag{B5}$$

and we see that again, as expected, with $l = l_0$, we have $\langle l_i l_j\rangle[l_j l_i] = S_{i+j}$. As we have massless legs some spinor products will consequentially vanish. In the two-mass case these are

$$\langle ll_2\rangle = 0, \quad \langle lK_2\rangle = 0, \quad [l_2K_2] = 0, \tag{B6}$$

and for the one-mass case

$$[l_1l_2] = 0, \quad \langle ll_2\rangle = 0, \quad \langle lK_2\rangle = 0, \quad \langle l_2K_2\rangle = 0, \quad [l_1K_3] = 0, \quad [l_2K_3] = 0, \tag{B7}$$

where K_3 is the momentum of the third leg.

The momentum parameterisation for the y_- solution is given in spinor components by

$$\langle l^-| = \frac{t}{[K_2K_1^\flat]}\langle K_2^+|\not{K}_1 + \frac{S_1}{\gamma}\langle\chi^-|, \qquad \langle l^+| = \frac{[\chi K_1^\flat]}{[K_2K_1^\flat]}\langle K_2^+|, \tag{B8}$$

and as a 4-vector by

$$l^\mu = \frac{[\chi K_1^\flat]}{2[K_2K_1^\flat]}\left(\frac{t}{[K_2K_1^\flat]}\langle K_2^+|\not{K}_1\gamma^\mu|K_2^-\rangle + \frac{S_1}{\gamma}\langle\chi^-|\gamma^\mu|K_2^-\rangle\right). \tag{B9}$$

The other momenta are given by

$$\langle l_1^-| = \frac{1}{[K_2K_1^\flat]}\langle K_2^+|\not{K}_1, \qquad \langle l_1^+| = -t\frac{[K_1^\flat\chi]}{[K_2K_1^\flat]}\langle K_2^+| - \langle K_1^{\flat,+}|,$$
$$\langle l_2^-| = \frac{1}{[\chi K_1^\flat]}\langle K_1^{\flat,+}|\not{K}_3 + \frac{t}{[K_2K_1^\flat]}\langle K_2^+|\not{K}_1, \qquad \langle l_2^+| = \frac{[\chi K_1^\flat]}{[K_2K_1^\flat]}\langle K_2^+|. \tag{B10}$$

The spinor products formed from these are given by

$$\begin{aligned}
\langle l l_1\rangle &= \frac{S_1}{\gamma}\langle\chi K_1^\flat\rangle, & [l l_1] &= [K_1^\flat\chi],\\
\langle l l_2\rangle &= \frac{[K_2 K_1^\flat]}{[\chi K_1^\flat]}\langle K_2 l\rangle, & [l l_2] &= 0,\\
\langle l_1 l_2\rangle &= \frac{S_3}{\gamma}\langle K_1^\flat\chi\rangle, & [l_1 l_2] &= [\chi K_1^\flat].
\end{aligned} \tag{B11}$$

and again $\langle l_i l_j\rangle[l_j l_i] = S_{i+j}$ as expected. The vanishing spinor products in the two mass case are

$$[l l_2] = 0, \qquad [l K_2] = 0, \qquad [l_2 K_2] = 0, \tag{B12}$$

and in the one mass case

$$\langle l_1 l_2\rangle = 0, \quad [l l_2] = 0, \quad [l K_2] = 0, \quad [l_2 K_2] = 0, \quad \langle l_1 K_3\rangle = 0, \quad \langle l_2 K_3\rangle = 0. \tag{B13}$$

APPENDIX C: THE SCALAR INTEGRAL FUNCTIONS

The scalar bubble integral with massive leg K_1 given in figure 6 is defined as

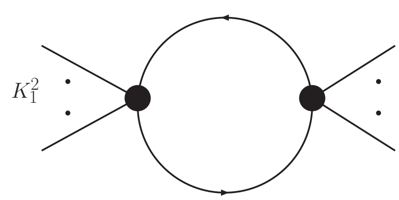

FIG. 6: The scalar bubble integral with a leg of mass K_1^2.

$$B_0(K_1^2) = (-i)(4\pi)^{2-\epsilon}\int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}}\frac{1}{l^2(l-K_1)^2}, \tag{C1}$$

and is given by

$$B_0(K_1^2) = \frac{r_\Gamma}{\epsilon(1-2\epsilon)}(-K_1^2)^{-\epsilon} = r_\Gamma\left(\frac{1}{\epsilon} - \ln(-K_1^2) + 2\right) + \mathcal{O}(\epsilon), \tag{C2}$$

with

$$r_\Gamma = \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}. \tag{C3}$$

The general form of the scalar triangle integral with the masses of its legs labelled K_1^2, K_2^2 and K_3^2 given in figure 7 is defined as

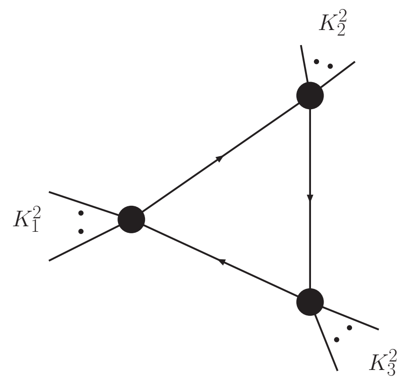

FIG. 7: The scalar triangle with its three legs of mass K_1^2, K_2^2 and K_3^2.

$$C_0(K_1^2, K_2^2) = i(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon} l}{(2\pi)^{4-2\epsilon}} \frac{1}{l^2(1-K_1)^2(l-K_2)^2}, \tag{C4}$$

and separates into three cases depending upon the masses of these external legs. In the one mass case we have $K_2^2 = 0$ and $K_3^2 = 0$ and the corresponding integral is given by

$$C_0(K_1^2, K_2^2) = \frac{r_\Gamma}{\epsilon^2}(-K_1^2)^{-1-\epsilon} = \frac{r_\Gamma}{(-K_1^2)}\left(\frac{1}{\epsilon^2} - \frac{\ln(-K_1^2)}{\epsilon} + \frac{\ln^2(-K_1^2)}{2}\right) + \mathcal{O}(\epsilon), \tag{C5}$$

If two legs are massive the integral, assuming $K_3^2 = 0$, is given by

$$\begin{aligned} C_0(K_1^2, K_2^2) &= \frac{r_\Gamma}{\epsilon^2} \frac{(-K_1^2)^{-\epsilon} - (-K_2^2)^{-\epsilon}}{(-K_1^2) - (-K_2^2)} \\ &= \frac{r_\Gamma}{(-K_1^2) - (-K_2^2)}\left(-\frac{\ln(-K_1^2) - \ln(-K_2^2)}{\epsilon} + \frac{\ln^2(-K_1^2) - \ln^2(-K_2^2)}{2}\right). \end{aligned} \tag{C6}$$

Finally if all three legs are massive then the integral is as given in [53, 54]

$$C_0(K_1^2, K_2^2) = \frac{i}{\sqrt{\Delta_3}} \sum_{j=1}^{3} \left[\mathrm{Li}_2\left(-\left(\frac{1+i\delta_j}{1-i\delta_j}\right)\right) - \mathrm{Li}_2\left(-\left(\frac{1-i\delta_j}{1+i\delta_j}\right)\right)\right] + \mathcal{O}(\epsilon), \tag{C7}$$

where

$$\begin{aligned} \delta_1 &= \frac{K_1^2 - K_2^2 - (K_1+K_2)^2}{\sqrt{\Delta_3}}, \\ \delta_1 &= \frac{-K_1^2 + K_2^2 - (K_1+K_2)^2}{\sqrt{\Delta_3}}, \\ \delta_1 &= \frac{-K_1^2 - K_2^2 + (K_1+K_2)^2}{\sqrt{\Delta_3}}, \end{aligned} \tag{C8}$$

and

$$\Delta_3 = -(K_2^2)^2 - (K_2^2)^2 - (K_3^2)^2 + 2(K_1^2 K_2^2 + K_3^2 K_1^2 + K_2^2 K_3^2) = -4\Delta, \tag{C9}$$

with Δ given by eq. (4.15).

The general form for a scalar box function is given by

$$D_0(K_1^2, K_2^2, K_3^2) = (-i)(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}l}{(2\pi)^{4-2\epsilon}} \frac{1}{l^2(l-K_1)^2(l-K_2)^2(l-K_3)^2}. \qquad \text{(C10)}$$

The solution of this integral is split up into classes depending upon the masses of the external legs. These solutions are labelled as zero mass I_4^{0m}, one mass I_4^{1m}, two mass hard, I_4^{2mh}, two mass easy I_4^{2me}, three mass I_4^{3m} and four mass I_4^{4m} integrals. The results for which can be found in the literature, for example in [17].

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